Integrable discretizations of the short pulse equation

Bao-Feng Feng[1], Ken-ichi Maruno[1] and Yasuhiro Ohta[2]

[1] Department of Mathematics, The University of Texas-Pan American, Edinburg, TX 78541
[2] Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan

E-mail: kmaruno@utpa.edu

Abstract
In this paper, we propose integrable semi-discrete and full-discrete analogues
of the short pulse (SP) equation. The key construction is the bilinear form
and determinant structure of solutions of the SP equation. We also give
the determinant formulas of N-soliton solutions of the semi-discrete and full-
discrete analogues of the SP equations, from which the multi-loop and multi-
breather solutions can be generated. In the continuous limit, the full-discrete SP
equation converges to the semi-discrete SP equation, and then to the continuous
SP equation. Based on the semi-discrete SP equation, an integrable numerical
scheme, i.e. a self-adaptive moving mesh scheme, is proposed and used for the
numerical computation of the short pulse equation.

PACS numbers: 02.30.Ik, 05.45.Yv, 42.65.Tg, 42.81.Dp

1. Introduction

Most recently, the short pulse (SP) equation

$$u_{xt} = u + \tfrac{1}{6}(u^3)_{xx} \qquad\qquad (1.1)$$

was derived as a model equation for the propagation of ultra-short optical pulses in nonlinear
media [1, 2]. Here, $u = u(x, t)$ represents the magnitude of the electric field, and the
subscripts t and x denote partial differentiation. Apart from the context of nonlinear optics,
the SP equation has also been derived as an integrable differential equation associated with
pseudospherical surfaces [3]. The SP equation has been shown to be completely integrable
[3–7]. The loop soliton solutions as well as smooth soliton solutions of the SP equation were
found in [8–10]. The connection between the SP equation and the sine-Gordon equation
through the hodograph transformation was clarified in [11], and then the N-soliton solutions,
including multi-loop and multi-breather soliton solutions, were given by using the Hirota
bilinear method.

Integrable discretizations of soliton equations have received considerable attention
recently [12–15]. In our recent work, we proposed an integrable semi-discrete analogue

of the Camassa–Holm (CH) equation and applied it as a numerical scheme, i.e. a self-adaptive moving mesh scheme [16, 17]. The key discretization is an introduction of an non-uniform mesh, which plays a role of the hodograph transformation as in the continuous case.

In this paper, we attempt to construct integrable semi-discrete and full-discretizations of the SP equation by the same approach used in the CH equation. We also attempt to use the semi-discrete analogue of the SP equation as a self-adaptive moving mesh scheme to perform numerical simulations.

The rest of the present communication is organized as follows. In section 2, we review the bilinear equations and determinant solutions of the SP equation. In section 3, we propose an integrable semi-discrete analogue of the SP equation, whose *N*-soliton solutions are also constructed in terms of determinant form. By using the semi-discrete analogue of the SP equation as a self-adaptive moving mesh scheme, the numerical results for one- and two-loop soliton solutions are also presented. In section 4, the full-discrete analogue of the SP equation is proposed. The communication is concluded by section 5.

2. Bilinear equations and determinant solutions of the short pulse equation

In this section, the results in [11], regarding the bilinear equations and the solutions of the SP equation, will be briefly reviewed.

First, by introducing the new dependent variable

$$r^2 = 1 + u_x^2, \tag{2.1}$$

the SP equation is rewritten as

$$r_t = \left(\tfrac{1}{2} u^2 r \right)_x. \tag{2.2}$$

Introducing the hodograph transformation

$$\mathrm{d}y = r \, \mathrm{d}x + \tfrac{1}{2} u^2 r \, \mathrm{d}t, \qquad \mathrm{d}s = \mathrm{d}t, \tag{2.3}$$

i.e.

$$\frac{\partial}{\partial t} = \frac{1}{2} u^2 r \frac{\partial}{\partial y} + \frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial x} = r \frac{\partial}{\partial y},$$

we obtain

$$r_s = r^2 u u_y, \tag{2.4}$$

where

$$r^2 = 1 + r^2 u_y^2.$$

Equation (2.4) can also be cast into a form of

$$\left(\frac{1}{r} \right)_s = - \left(\frac{1}{2} u^2 \right)_y. \tag{2.5}$$

Introducing new variables

$$r = \frac{1}{\cos \phi}, \qquad u = \phi_s, \tag{2.6}$$

equation (2.5) leads to the sine-Gordon equation

$$\phi_{ys} = \sin \phi. \tag{2.7}$$

Moreover, as is shown in [18, 22], upon the dependent variable transformation

$$\phi(y, s) = 2\mathrm{i} \ln \frac{F^*(y, s)}{F(y, s)},$$

the sine-Gordon equation (2.7) leads to the following bilinear equations:

$$FF_{ys} - F_y F_s = \tfrac{1}{4}(F^2 - F^{*2}), \tag{2.8}$$

$$F^* F^*_{ys} - F^*_y F^*_s = \tfrac{1}{4}(F^{*2} - F^2), \tag{2.9}$$

where F^* is the complex conjugate of F. Henceforth, the solutions of the SP equation are obtained by F and F^* through the dependent variable transformation

$$u(y, s) = \frac{\partial}{\partial s}\phi(y, s) = \frac{\partial}{\partial s}\left(2\mathrm{i}\ln\frac{F^*(y, s)}{F(y, s)}\right). \tag{2.10}$$

In what follows, we will show that the bilinear equations (2.8) and (2.9) are actually obtained as the 2-reduction of the two-dimensional Toda lattice (2DTL) equations [19–22]:

$$\tfrac{1}{2}D_Y D_S \tau_n \cdot \tau_n = \tau_n{}^2 - \tau_{n+1}\tau_{n-1}, \tag{2.11}$$

i.e.,

$$\tau_n \frac{\partial^2 \tau_n}{\partial Y \partial S} - \frac{\partial \tau_n}{\partial Y}\frac{\partial \tau_n}{\partial S} = \tau_n{}^2 - \tau_{n+1}\tau_{n-1}, \tag{2.12}$$

where D_x is the Hirota D-operator which is defined as

$$D_x^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^n f(x)g(y)|_{y=x}.$$

Applying the 2-reduction $\tau_{n-1} = \alpha^{-1}\tau_{n+1}$ (α is a constant), we obtain

$$\tau_n \frac{\partial^2 \tau_n}{\partial Y \partial S} - \frac{\partial \tau_n}{\partial Y}\frac{\partial \tau_n}{\partial S} = \tau_n{}^2 - \tau_{n+1}^2, \tag{2.13}$$

where the gauge transformation $\tau_n \to \alpha^{\frac{n}{2}}\tau_n$ is used. Letting $f = \tau_0$ and $\bar{f} = \tau_1$, we have

$$ff_{YS} - f_Y f_S = f^2 - \bar{f}^2, \tag{2.14}$$

$$\bar{f}\bar{f}_{YS} - \bar{f}_Y \bar{f}_S = \bar{f}^2 - f^2. \tag{2.15}$$

Under the independent variable transformation $y = 2Y$, $s = 2S$, we obtain

$$ff_{ys} - f_y f_s = \tfrac{1}{4}(f^2 - \bar{f}^2), \tag{2.16}$$

$$\bar{f}\bar{f}_{ys} - \bar{f}_y \bar{f}_s = \tfrac{1}{4}(\bar{f}^2 - f^2), \tag{2.17}$$

which are bilinear equations of the SP equation.

Next, we give the Casorati determinant (*N*-soliton) solution of the SP equation. It is known that the Casorati determinant solution of the 2DTL equation is of the form [21, 22]

$$\tau_n(Y, S) = \left|\psi_i^{(n+j-1)}(Y, S)\right|_{1\leqslant i, j\leqslant N}, \tag{2.18}$$

where $\psi_i^{(n)}(Y, S)$ satisfies linear dispersion relations

$$\frac{\partial \psi_i^{(n)}}{\partial Y} = \psi_i^{(n+1)}, \qquad \frac{\partial \psi_i^{(n)}}{\partial S} = \psi_i^{(n-1)}. \tag{2.19}$$

For example, a particular choice of $\psi_i^{(n)}(Y, S)$

$$\psi_i^{(n)}(Y, S) = c_{i,1}p_i^n\,\mathrm{e}^{p_i Y + \frac{1}{p_i}S + \eta_{0i}} + c_{i,2}q_i^n\,\mathrm{e}^{q_i Y + \frac{1}{q_i}S + \eta'_{0i}}, \tag{2.20}$$

with $c_{i,1}$ and $c_{i,2}$ being constants, satisfies the linear dispersion relations and gives the *N*-soliton solutions.

Applying the 2-reduction $q_i = -p_i$ and the change of variables $y = 2Y$ and $s = 2S$, we obtain the determinant solution of bilinear equations (2.16) and (2.17):

$$f(y, s) = \tau_0(y, s), \quad \bar{f}(y, s) = \tau_1(y, s),$$

$$\tau_n(y, s) = \left| \psi_i^{(n+j-1)}(y, s) \right|_{1 \leqslant i, j \leqslant N}, \tag{2.21}$$

where

$$\psi_i^{(n)}(y, s) = c_{i,1} p_i^n \, \mathrm{e}^{\frac{1}{2} p_i y + \frac{1}{2p_i} s + \eta_{0i}} + c_{i,2}(-p_i)^n \, \mathrm{e}^{-\frac{1}{2} p_i y - \frac{1}{2p_i} s + \eta'_{0i}}. \tag{2.22}$$

Since u is real and the dependent variable transformation of u includes the imaginary number, we must consider the reality condition of u. Let us introduce α and β such that $F^* = \alpha \bar{f}$ and $F = \beta f$, where F and F^* are complex conjugate of each other. Note that F and F^* also satisfy the bilinear equations (2.16) and (2.17) because of

$$u = \frac{\partial}{\partial s} \left(2\mathrm{i} \ln \frac{F^*}{F} \right) = \frac{\partial}{\partial s} \left(2\mathrm{i} \ln \frac{\alpha \bar{f}}{\beta f} \right) = \frac{\partial}{\partial s} \left(2\mathrm{i} \ln \frac{\bar{f}}{f} + 2\mathrm{i} \ln \frac{\alpha}{\beta} \right) = \frac{\partial}{\partial s} \left(2\mathrm{i} \ln \frac{\bar{f}}{f} \right). \tag{2.23}$$

Thus, a set of F and F^* gives solutions of the SP equation as well as a set of f and $\bar{f}$. By choosing phase constants appropriately, the functions f and $\bar{f}$ can be made to be complex conjugate of each other to keep the reality and regularity of u. For example, the following choice:

$$\psi_i^{(n)} = p_i^n \, \mathrm{e}^{\frac{1}{2} p_i y + \frac{1}{2p_i} s + \eta_{0i} - \mathrm{i}\pi/4} + (-p_i)^n \, \mathrm{e}^{-\frac{1}{2} p_i y - \frac{1}{2p_i} s + \eta'_{0i} + \mathrm{i}\pi/4} \tag{2.24}$$

guarantees the reality and regularity of the solution.

Summarizing the above results, the determinant (*N*-soliton) solution of the SP equation is given by

$$u(y, s) = \frac{\partial}{\partial s} \left(2\mathrm{i} \ln \frac{\bar{f}(y, s)}{f(y, s)} \right), \tag{2.25}$$

$$x = y - 2(\ln \bar{f} f)_s, \qquad t = s,$$

$$f(y, s) = \tau_0(y, s), \qquad \bar{f}(y, s) = \tau_1(y, s),$$

$$\tau_n(y, s) = \left| \psi_i^{(n+j-1)}(y, s) \right|_{1 \leqslant i, j \leqslant N},$$

where

$$\psi_i^{(n)} = p_i^n \, \mathrm{e}^{\frac{1}{2} p_i y + \frac{1}{2p_i} s + \eta_{0i} - \mathrm{i}\pi/4} + (-p_i)^n \, \mathrm{e}^{-\frac{1}{2} p_i y - \frac{1}{2p_i} s + \eta'_{0i} + \mathrm{i}\pi/4}.$$

3. An integrable semi-discretization of the short pulse equation and numerical computations

Based on the above fact, we construct the integrable spatial-discretization of the SP equation. Consider the following Casorati determinant:

$$\tau_n(k, S) = \left| \psi_i^{(n+j-1)}(k, S) \right|_{1 \leqslant i, j \leqslant N}, \tag{3.1}$$

where $\psi_i^{(n)}$ satisfies the dispersion relations

$$\Delta_k \psi_i^{(n)} = \psi_i^{(n+1)}, \tag{3.2}$$

$$\partial_S \psi_i^{(n)} = \psi_i^{(n-1)}. \tag{3.3}$$

Here Δ_k is the backward difference operator with the spacing constant a:

$$\Delta_k f(k) = \frac{f(k) - f(k-1)}{a}.$$

Particularly, one can choose

$$\psi_i^{(n)}(k, S) = c_{i,1} p_i^n (1 - a p_i)^{-k} \, \mathrm{e}^{\frac{1}{p_i} S + \xi_{i0}} + c_{i,2} q_i^n (1 - a q_i)^{-k} \, \mathrm{e}^{\frac{1}{q_i} S + \eta_{i0}}, \tag{3.4}$$

which automatically satisfies the dispersion relations (3.2) and (3.3). The above Casorati determinant satisfies the bilinear form of the semi-discrete 2DTL equation (the Bäcklund transformation of the bilinear equation of 2DTL equation) [22, 23]

$$\left(\frac{1}{a} D_S - 1\right) \tau_n(k+1) \cdot \tau_n(k) + \tau_{n+1}(k+1) \tau_{n-1}(k) = 0. \tag{3.5}$$

Applying the 2-reduction

$$q_i = -p_i,$$

and letting

$$f_k = \tau_0(k), \qquad \bar{f}_k = \tau_1(k) = \left(\prod_{i=1}^N p_i^2\right) \tau_{-1}(k),$$

we obtain

$$\frac{1}{a} D_S f_{k+1} \cdot f_k - f_{k+1} f_k + \bar{f}_{k+1} \bar{f}_k = 0, \tag{3.6}$$

$$\frac{1}{a} D_S \bar{f}_{k+1} \cdot \bar{f}_k - \bar{f}_{k+1} \bar{f}_k + f_{k+1} f_k = 0, \tag{3.7}$$

where the gauge transformation $\tau_n \to \left(\prod_{i=1}^N p_i\right)^n \tau_n$ is used. Note that f_k and $\bar{f}_k$ can be made complex conjugate of each other by choosing the phase constants properly. Under the change of independent variable $s = 2S$, equation (3.5) implies the following two bilinear equations:

$$\frac{2}{a} D_s f_{k+1} \cdot f_k - f_{k+1} f_k + \bar{f}_{k+1} \bar{f}_k = 0, \tag{3.8}$$

$$\frac{2}{a} D_s \bar{f}_{k+1} \cdot \bar{f}_k - \bar{f}_{k+1} \bar{f}_k + f_{k+1} f_k = 0, \tag{3.9}$$

which can be readily shown to be equivalent to

$$-\left(\frac{2}{a} \left(\ln \frac{f_{k+1}}{f_k}\right)_s - 1\right) = \frac{\bar{f}_{k+1} \bar{f}_k}{f_{k+1} f_k}, \tag{3.10}$$

$$-\left(\frac{2}{a} \left(\ln \frac{\bar{f}_{k+1}}{\bar{f}_k}\right)_s - 1\right) = \frac{f_{k+1} f_k}{\bar{f}_{k+1} \bar{f}_k}, \tag{3.11}$$

where the subscript 's' denotes the derivative with respect to the continuous variable 's'.

Subtracting the above two equations, one obtains

$$\frac{2}{a}\left(\left(\ln \frac{\bar{f}_{k+1}}{\bar{f}_k}\right)_s - \left(\ln \frac{f_{k+1}}{f_k}\right)_s\right) = \frac{\bar{f}_{k+1} \bar{f}_k}{f_{k+1} f_k} - \frac{f_{k+1} f_k}{\bar{f}_{k+1} \bar{f}_k}. \tag{3.12}$$

Introducing the dependent variable transformation $\phi_k(s) = 2i \ln\left(\frac{\bar{f}_k(s)}{f_k(s)}\right)$, one arrives at

$$\frac{1}{2a}\left(\frac{d\phi_{k+1}}{ds} - \frac{d\phi_k}{ds}\right) = \sin\left(\frac{\phi_{k+1} + \phi_k}{2}\right), \tag{3.13}$$

which is nothing but an integrable semi-discretization of the sine-Gordon equation. Note that this is also known as the Bäcklund transformation of the sine-Gordon equation [24, 25].

It is obvious that, from the semi-discrete sine-Gordon equation (3.13), the equation

$$\left(\cos\left(\frac{\phi_{k+1} + \phi_k}{2}\right)\right)_s = -\frac{1}{4a}\left(\left(\frac{d\phi_{k+1}}{ds}\right)^2 - \left(\frac{d\phi_k}{ds}\right)^2\right) \tag{3.14}$$

is implied. By introducing the variable transformations

$$u_k = \frac{d\phi_k}{ds} = \frac{d}{ds}\left(2i \ln \frac{\bar{f}_k(s)}{f_k(s)}\right), \qquad \delta_k = a \cos\left(\frac{\phi_{k+1} + \phi_k}{2}\right), \tag{3.15}$$

it then follows that

$$\frac{d\delta_k}{ds} = -\frac{u_{k+1}^2 - u_k^2}{4}, \tag{3.16}$$

which is the first equation of a semi-discrete analogue of the SP equation.

From the facts

$$\cos^2\left(\frac{\phi_{k+1} + \phi_k}{2}\right) + \sin^2\left(\frac{\phi_{k+1} + \phi_k}{2}\right) = 1, \tag{3.17}$$

$$\sin\left(\frac{\phi_{k+1} + \phi_k}{2}\right) = \frac{u_{k+1} - u_k}{2a}, \tag{3.18}$$

and

$$\frac{1}{r_k} = \frac{\delta_k}{a} = \cos\left(\frac{\phi_{k+1} + \phi_k}{2}\right), \tag{3.19}$$

it follows that

$$\frac{\delta_k^2}{a^2} + \frac{(u_{k+1} - u_k)^2}{4a^2} = 1,$$

i.e.

$$\delta_k^2 = a^2 - \frac{(u_{k+1} - u_k)^2}{4}, \tag{3.20}$$

which becomes another equation of a semi-discrete analogue of the SP equation.

Summarizing the above results, we obtained an integrable semi-discrete analogue of the SP equation

$$(u_{k+1} - u_k)^2 = 4(a^2 - \delta_k^2), \tag{3.21}$$

$$\frac{d\delta_k}{ds} = -\frac{u_{k+1}^2 - u_k^2}{4}, \tag{3.22}$$

where the x-coordinate of the k-th lattice point is given by $X_k = X_0 + \sum_{l=0}^{k-1} \delta_l = ka - \frac{d}{ds}(\ln \bar{f}_k f_k)$. From the construction, the semi-discrete analogue of the SP equation

has the following Casorati determinant solution:

$$u_k(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(2\mathrm{i}\ln\frac{\bar{f}_k}{f_k}\right), \qquad \delta_k = \frac{a}{2}\left(\frac{\bar{f}_{k+1}\bar{f}_k}{f_{k+1}f_k} + \frac{f_{k+1}f_k}{\bar{f}_{k+1}\bar{f}_k}\right),$$

$$X_k = ka - \frac{\mathrm{d}}{\mathrm{d}s}(\ln\bar{f}_k\bar{f}_k),$$

$$f_k(s) = \tau_0(k,s), \qquad \bar{f}_k(s) = \tau_1(k,s),$$

$$\tau_n(k,s) = \left|\psi_i^{(n+j-1)}(k,s)\right|_{1\leqslant i,j\leqslant N},$$

(3.23)

with

$$\psi_i^{(n)}(k,s) = p_i^n(1-ap_i)^{-k}\,\mathrm{e}^{\frac{1}{2p_i}s+\xi_{i0}-\mathrm{i}\pi/4} + (-p_i)^n(1+ap_i)^{-k}\,\mathrm{e}^{-\frac{1}{2p_i}s+\eta_{i0}+\mathrm{i}\pi/4},$$

where the phase constants $\pm\mathrm{i}\pi/4$ play the role of keeping the reality and regularity of the solution.

Note that a^2 must be always greater than or equal to δ_k^2 because $(u_{k+1}-u_k)^2\geqslant 0$. This can be easily verified by

$$|\delta_k| = \left|a\cos\left(\frac{\phi_{k+1}+\phi_k}{2}\right)\right| \leqslant |a|.$$

(3.24)

The mesh size of self-adaptive mesh $|\delta_k|$ is always chosen as less than $|a|$.

We can rewrite the semi-discrete SP equation in an alternative form which converges to the SP equation in the continuous limit $\delta_k\to 0$. Multiplying equation (3.22) by $2\delta_k$, we have

$$\frac{\mathrm{d}\delta_k^2}{\mathrm{d}s} = -\delta_k\frac{u_{k+1}^2-u_k^2}{2}.$$

(3.25)

A substitution of δ_k^2 from equation (3.21) into equation (3.25) leads to

$$\frac{\mathrm{d}(u_{k+1}-u_k)}{\mathrm{d}s} = \delta_k(u_{k+1}+u_k).$$

(3.26)

Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{u_{k+1}-u_k}{\delta_k}\right) = \frac{1}{\delta_k}\frac{\mathrm{d}(u_{k+1}-u_k)}{\mathrm{d}s} - \frac{u_{k+1}-u_k}{\delta_k^2}\frac{\mathrm{d}\delta_k}{\mathrm{d}s},$$

(3.27)

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{u_{k+1}-u_k}{\delta_k}\right) = u_{k+1}+u_k + \frac{u_{k+1}+u_k}{4}\left(\frac{u_{k+1}-u_k}{\delta_k}\right)^2,$$

(3.28)

by using equations (3.26) and (3.22). Equation (3.28) gives another form of the semi-discrete SP equation. In the continuous limit $a\to 0$ ($\delta_k\to 0$), we have

$$\frac{u_{k+1}-u_k}{\delta_k} \to \frac{\mathrm{d}u}{\mathrm{d}X}, \qquad \frac{u_{k+1}+u_k}{2} \to u,$$

$$\frac{\partial X}{\partial s} = \frac{\partial X_0}{\partial s} + \sum_{j=0}^{k-1}\frac{\mathrm{d}\delta_j}{\mathrm{d}s} = -\frac{1}{4}\sum_{j=0}^{k-1}\left(u_{j+1}^2-u_j^2\right) \to -\frac{1}{4}u^2,$$

$$\partial_s = \partial_t + \frac{\partial X}{\partial s}\partial_X \to \partial_t - \frac{1}{4}u^2\partial_X,$$

Consequently, equation (3.28) converges to

$$\left(\partial_t - \tfrac{1}{4}u^2\partial_X\right)u_x = 2u + \tfrac{1}{2}uu_X^2.$$

By the scaling transformation $x=2X$, one arrives at
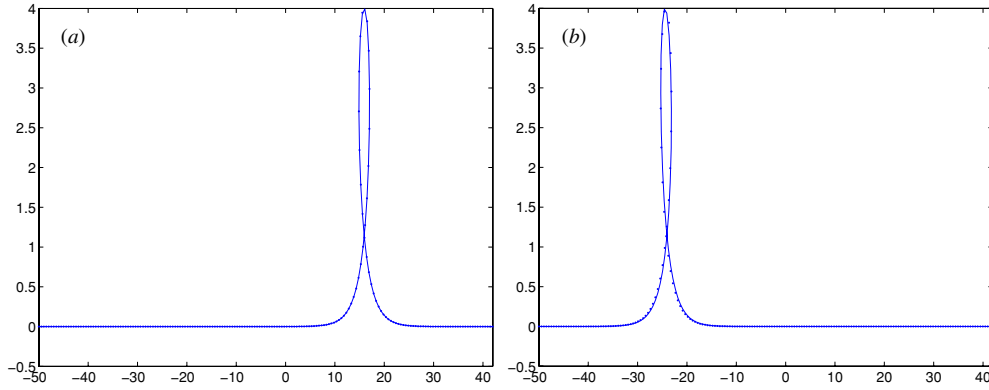
$$u_{xt} = u + uu_x^2 + \tfrac{1}{2}u^2u_{xx},$$

7

Figure 1. Numerical solution for one-loop soliton solution with $p_1 = 0.5$ at (*a*) $t = 0.0$; (*b*) $t = 10.0$, dot line: numerical solution; solid line: exact solution.

which turns out to be the SP equation

$$u_{xt} = u + \tfrac{1}{6}(u^3)_{xx}.$$

In a similar way employed in [16, 17], the semi-discrete analogue of the SP equation can be used as a novel numerical scheme, i.e. the so-called self-adaptive moving mesh method, to perform numerical computations for the SP equation. However, the first equation (3.21) has ambiguity for determining the sign even if the non-uniform mesh δ_k is solved from the second equation (3.22). To avoid this difficulty, we introduce an intermediate variable $\bar{\phi}_k = (\phi_{k+1} + \phi_k)/2$, and employ the following scheme:

$$\begin{cases} u_{k+1} - u_k = 2a \sin(\bar{\phi}_k), \\ \dfrac{\mathrm{d}\bar{\phi}_k}{\mathrm{d}s} = \dfrac{u_{k+1} + u_k}{2}, \end{cases} \tag{3.29}$$

which can be derived from equations (3.18) and (3.15). Equations (3.29) are equivalent to the integrable semi-discrete analogue of the SP equation, and the relation between the non-uniform mesh δ_k and $\bar{\phi}_k$ is $\delta_k = a \cos(\bar{\phi}_k)$. Figures 1 and 2 are numerical results for one-loop and two-loop soliton solutions, respectively. The time stepsize is $\Delta t = 0.01$, the number of grid points is $N = 200$, and the value of the spacing parameter is $a = 0.5$. The detailed numerical results by using the integrable semi-discrete SP equation will be reported somewhere else.

4. Full-discretizations of the short pulse equation

To construct a full-discrete analogue of the SP equation, we introduce one more discrete variable l which corresponds to the discrete time variable.

It is known that the τ-function

$$\tau_n(k, l) = \left| \psi_i^{(n+j-1)}(k, l) \right|_{1 \leqslant i, j \leqslant N}, \tag{4.1}$$

with

$$\psi_i^{(n)}(k, l) = c_{i,1} p_i^n (1 - ap_i)^{-k} \left(1 - b\frac{1}{p_i}\right)^{-l} \mathrm{e}^{\frac{1}{2p_i} s + \xi_{i0}}$$

$$+ c_{i,2} q_i^n (1 - aq_i)^{-k} \left(1 - b\frac{1}{q_i}\right)^{-l} \mathrm{e}^{\frac{1}{2q_i} s + \eta_{i0}},$$
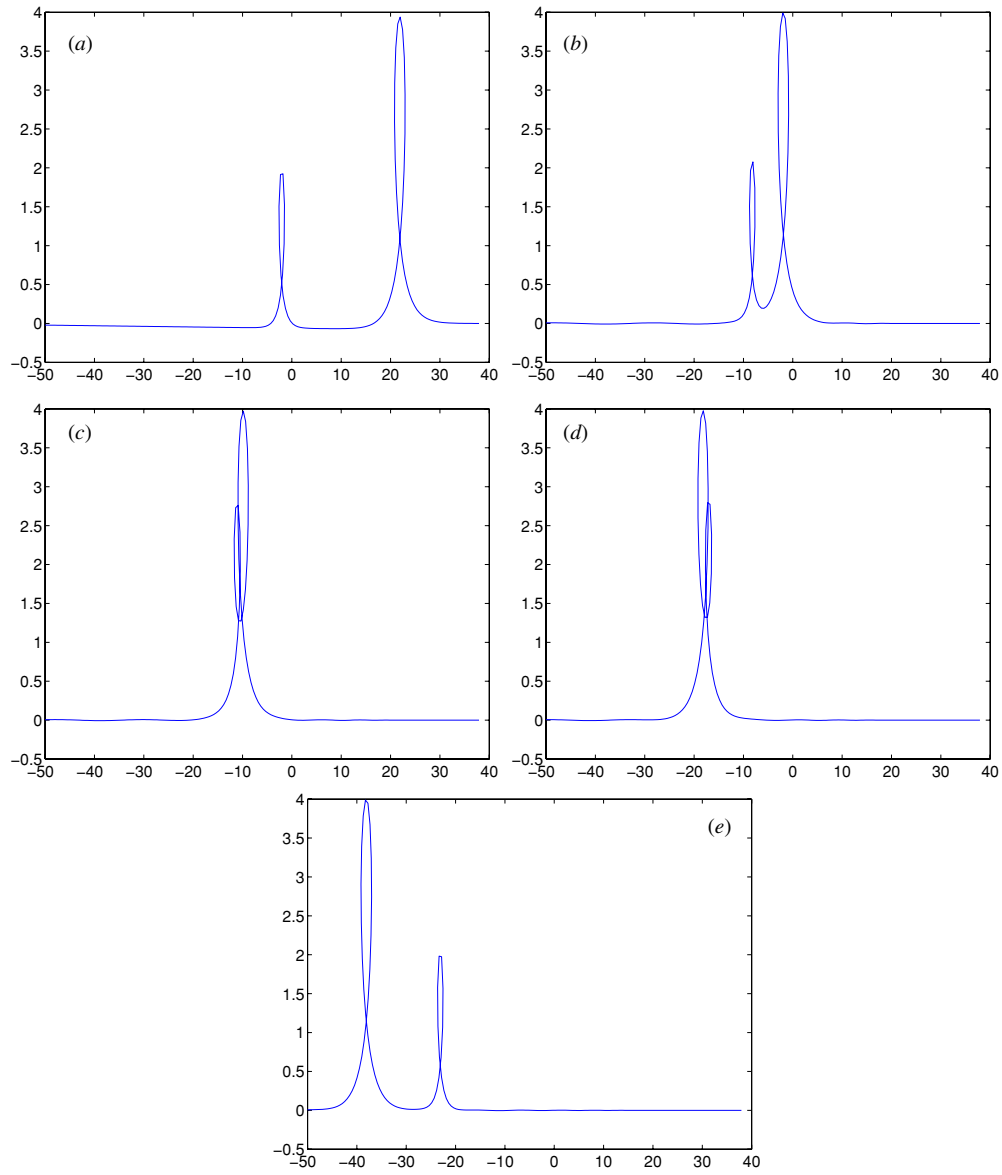
Figure 2. Numerical solutions for the collision of two-loop soliton solution with $p_1 = 0.5$, $p_2 = 1.0$ at (*a*) $t = 0.0$; (*b*) $t = 6.0$; (*c*) $t = 8.0$; (*d*) $t = 10.0$; (*e*) $t = 15.0$.

satisfies bilinear equations [23]

$$\left(\frac{2}{a}D_s - 1\right)\tau_n(k+1,l) \cdot \tau_n(k,l) + \tau_{n+1}(k+1,l)\tau_{n-1}(k,l) = 0, \tag{4.2}$$

and

$$(2bD_s - 1)\tau_n(k,l+1) \cdot \tau_{n+1}(k,l) + \tau_n(k,l)\tau_{n+1}(k,l+1) = 0. \tag{4.3}$$

Applying the 2-reduction $\tau_{n-1} = \left(\prod_{i=1}^{N} p_i^2 \right)^{-1} \tau_{n+1}$, i.e. adding constraints $q_i = -p_i$ to the N-soliton solution, we obtain

$$\left(\frac{2}{a} D_s - 1 \right) \tau_n(k+1, l) \cdot \tau_n(k, l) + \tau_{n+1}(k+1, l) \tau_{n+1}(k, l) = 0 \qquad (4.4)$$

and

$$(2b D_s - 1) \tau_n(k, l+1) \cdot \tau_{n+1}(k, l) + \tau_n(k, l) \tau_{n+1}(k, l+1) = 0, \qquad (4.5)$$

where the gauge transformation $\tau_n \rightarrow \left(\prod_{i=1}^{N} p_i \right)^n \tau_n$ is used. Letting

$$f_{k,l} = \tau_0(k, l), \qquad \bar{f}_{k,l} = \tau_1(k, l),$$

the bilinear equations (4.4) and (4.5) imply the following four equations:

$$\left(\frac{2}{a} D_s - 1 \right) f_{k+1,l} \cdot f_{k,l} + \bar{f}_{k+1,l} \bar{f}_{k,l} = 0, \qquad (4.6)$$

$$\left(\frac{2}{a} D_s - 1 \right) \bar{f}_{k+1,l} \cdot \bar{f}_{k,l} + f_{k+1,l} f_{k,l} = 0, \qquad (4.7)$$

$$(2b D_s - 1) f_{k,l+1} \cdot \bar{f}_{k,l} + f_{k,l} \bar{f}_{k,l+1} = 0, \qquad (4.8)$$

$$(2b D_s - 1) \bar{f}_{k,l+1} \cdot f_{k,l} + \bar{f}_{k,l} f_{k,l+1} = 0, \qquad (4.9)$$

which are actually equivalent to

$$\frac{2}{a} \left(\ln \frac{f_{k+1,l}}{f_{k,l}} \right)_s - 1 + \frac{\bar{f}_{k+1,l} \bar{f}_{k,l}}{f_{k+1,l} f_{k,l}} = 0, \qquad (4.10)$$

$$\frac{2}{a} \left(\ln \frac{\bar{f}_{k+1,l}}{\bar{f}_{k,l}} \right)_s - 1 + \frac{f_{k+1,l} f_{k,l}}{\bar{f}_{k+1,l} \bar{f}_{k,l}} = 0, \qquad (4.11)$$

$$2b \left(\ln \frac{f_{k,l+1}}{\bar{f}_{k,l}} \right)_s - 1 + \frac{f_{k,l} \bar{f}_{k,l+1}}{f_{k,l+1} \bar{f}_{k,l}} = 0, \qquad (4.12)$$

$$2b \left(\ln \frac{\bar{f}_{k,l+1}}{f_{k,l}} \right)_s - 1 + \frac{\bar{f}_{k,l} f_{k,l+1}}{\bar{f}_{k,l+1} f_{k,l}} = 0, \qquad (4.13)$$

where the subscript 's' denotes the derivative with respect to a continuous parameter 's'.

Note that f and $\bar{f}$ can be made complex conjugate of each other by choosing the phase constants properly. By introducing

$$u_{k,l} = \left(2i \ln \frac{\bar{f}_{k,l}}{f_{k,l}} \right)_s, \qquad (4.14)$$

and

$$X_{k,l} = ka - (\ln \bar{f}_{k,l} f_{k,l})_s, \qquad (4.15)$$

where $X_{k,l}$ is the x-coordinate of the kth lattice point at time l, we find the following relations:

$$u_{k+1,l} - u_{k,l} = ia \left(\frac{\bar{f}_{k+1,l} \bar{f}_{k,l}}{f_{k+1,l} f_{k,l}} - \frac{f_{k+1,l} f_{k,l}}{\bar{f}_{k+1,l} \bar{f}_{k,l}} \right), \qquad (4.16)$$

$$u_{k,l+1} + u_{k,l} = \frac{i}{b} \left(\frac{f_{k,l} \bar{f}_{k,l+1}}{f_{k,l+1} \bar{f}_{k,l}} - \frac{\bar{f}_{k,l} f_{k,l+1}}{\bar{f}_{k,l+1} f_{k,l}} \right), \qquad (4.17)$$

$$X_{k+1,l} - X_{k,l} = \frac{a}{2}\left(\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} + \frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}\right), \tag{4.18}$$

$$X_{k,l+1} - X_{k,l} = -\frac{1}{b} + \frac{1}{2b}\left(\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} + \frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}}\right). \tag{4.19}$$

It is straightforward to derive

$$(u_{k+1,l} - u_{k,l})^2 = 4\left(a^2 - \delta_{k,l}^2\right) \tag{4.20}$$

from equations (4.16) and (4.18), and

$$(u_{k,l+1} + u_{k,l})^2 = 4\left(\frac{1}{b^2} - \left(X_{k,l+1} - X_{k,l} + \frac{1}{b}\right)^2\right) \tag{4.21}$$

from equations (4.17) and (4.19), where $\delta_{k,l} = X_{k+1,l} - X_{k,l}$. Equations (4.20) and (4.21) give a full-discrete analogue of the SP equation.

Let us consider another full-discrete analogue of the SP equation. Using equations (4.16)–(4.19), we obtain

$$\frac{\bar{f}_{k+1,l}\,\bar{f}_{k,l}}{f_{k+1,l}\,f_{k,l}} = \frac{1}{a}\left(X_{k+1,l} - X_{k,l} - \mathrm{i}\frac{u_{k+1,l} - u_{k,l}}{2}\right), \tag{4.22}$$

$$\frac{f_{k+1,l}\,f_{k,l}}{\bar{f}_{k+1,l}\,\bar{f}_{k,l}} = \frac{1}{a}\left(X_{k+1,l} - X_{k,l} + \mathrm{i}\frac{u_{k+1,l} - u_{k,l}}{2}\right), \tag{4.23}$$

$$\frac{f_{k,l}\,\bar{f}_{k,l+1}}{f_{k,l+1}\,\bar{f}_{k,l}} = b\left(X_{k,l+1} - X_{k,l} + \frac{1}{b} - \mathrm{i}\frac{u_{k,l+1} + u_{k,l}}{2}\right), \tag{4.24}$$

$$\frac{\bar{f}_{k,l}\,f_{k,l+1}}{\bar{f}_{k,l+1}\,f_{k,l}} = b\left(X_{k,l+1} - X_{k,l} + \frac{1}{b} + \mathrm{i}\frac{u_{k,l+1} + u_{k,l}}{2}\right). \tag{4.25}$$

From relations (4.22)–(4.25), we have

$$\frac{X_{k+1,l+1} - X_{k,l+1} - \mathrm{i}\frac{u_{k+1,l+1} - u_{k,l+1}}{2}}{X_{k+1,l} - X_{k,l} - \mathrm{i}\frac{u_{k+1,l} - u_{k,l}}{2}} = \frac{X_{k+1,l+1} - X_{k+1,l} + \frac{1}{b} - \mathrm{i}\frac{u_{k+1,l+1} + u_{k+1,l}}{2}}{X_{k,l+1} - X_{k,l} + \frac{1}{b} + \mathrm{i}\frac{u_{k,l+1} + u_{k,l}}{2}}. \tag{4.26}$$

Equating the real part and imaginary part respectively, we have

$$(X_{k+1,l+1} - X_{k,l+1})\left(X_{k,l+1} - X_{k,l} + \frac{1}{b}\right) + \frac{u_{k+1,l+1} - u_{k,l+1}}{2}\frac{u_{k,l+1} + u_{k,l}}{2}$$

$$= \left(X_{k+1,l+1} - X_{k+1,l} + \frac{1}{b}\right)(X_{k+1,l} - X_{k,l}) - \frac{u_{k+1,l+1} + u_{k+1,l}}{2}\frac{u_{k+1,l} - u_{k,l}}{2}, \tag{4.27}$$

$$\left(X_{k,l+1} - X_{k,l} + \frac{1}{b}\right)(u_{k+1,l+1} - u_{k,l+1}) - (X_{k+1,l+1} - X_{k,l+1})(u_{k,l+1} + u_{k,l})$$

$$= \left(X_{k+1,l+1} - X_{k+1,l} + \frac{1}{b}\right)(u_{k+1,l} - u_{k,l}) + (X_{k+1,l} - X_{k,l})(u_{k+1,l+1} + u_{k+1,l}), \tag{4.28}$$

which can be rearranged into the following simpler form:

$$(X_{k+1,l+1} - X_{k+1,l} - X_{k,l+1} + X_{k,l})\left(\frac{1}{b} - X_{k+1,l} + X_{k,l+1}\right)$$

$$= -\frac{u_{k+1,l+1} + u_{k+1,l} - u_{k,l+1} - u_{k,l}}{2}\frac{u_{k+1,l} + u_{k,l+1}}{2}, \tag{4.29}$$

$$(u_{k+1,l+1} - u_{k+1,l} - u_{k,l+1} + u_{k,l}) \left(\frac{2}{b} + X_{k+1,l+1} - X_{k+1,l} + X_{k,l+1} - X_{k,l} \right)$$
$$= (X_{k+1,l+1} + X_{k+1,l} - X_{k,l+1} - X_{k,l})(u_{k+1,l+1} + u_{k+1,l} + u_{k,l+1} + u_{k,l}). \qquad (4.30)$$

Equations (4.29) and (4.30) constitute another form of integrable full-discretization of the SP equation. Taking the continuous limit $b \to 0$ in time, we obtain

$$(X_{k+1} - X_k)_s = -\tfrac{1}{4}(u_{k+1} - u_k)(u_{k+1} + u_k), \qquad (4.31)$$

and

$$(u_{k+1} - u_k)_s = (X_{k+1} - X_k)(u_{k+1} + u_k), \qquad (4.32)$$

which are nothing but the semi-discrete analogue of the SP equation (3.21) and (3.22). Here we used $\frac{F_{l+1}-F_l}{2b} \to \partial_s F$ as $b \to 0$.

From the construction of the full-discrete analogue of the SP equation, the determinant solution of the full-discrete SP equation is

$$u_{k,l} = \mathrm{i} \left(\frac{\bar{g}_{k,l}}{\bar{f}_{k,l}} - \frac{g_{k,l}}{f_{k,l}} \right) = \frac{\mathrm{d}}{\mathrm{d}s} \left(2\mathrm{i} \ln \frac{\bar{f}_{k,l}}{f_{k,l}} \right), \qquad (4.33)$$

$$X_{k,l} = ka - \frac{1}{2} \left(\frac{\bar{g}_{k,l}}{\bar{f}_{k,l}} + \frac{g_{k,l}}{f_{k,l}} \right) = ka - \frac{\mathrm{d}}{\mathrm{d}s} (\ln \bar{f}_{k,l} f_{k,l}), \qquad (4.34)$$

$$f_{k,l} = \tau_0(k,l), \qquad \bar{f}_{k,l} = \tau_1(k,l),$$
$$g_{k,l} = \rho_0(k,l), \qquad \bar{g}_{k,l} = \rho_1(k,l),$$

$$\tau_n(k,l) = \begin{vmatrix} \psi_1^{(n)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \cdots & \vdots & \vdots \\ \psi_N^{(n)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l) \end{vmatrix} \qquad \text{and}$$

$$\rho_n(k,l) = \begin{vmatrix} \psi_1^{(n-1)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n-1)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \cdots & \vdots & \vdots \\ \psi_N^{(n-1)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l) \end{vmatrix},$$

with

$$\psi_i^{(n)}(k,l) = p_i^n (1 - ap_i)^{-k} \left(1 - b\frac{1}{p_i} \right)^{-l} \mathrm{e}^{\frac{1}{2p_i}s + \xi_{i0} - \mathrm{i}\pi/4}$$
$$+ (-p_i)^n (1 + ap_i)^{-k} \left(1 + b\frac{1}{p_i} \right)^{-l} \mathrm{e}^{-\frac{1}{2p_i}s + \eta_{i0} + \mathrm{i}\pi/4},$$

where the phase constants $\pm \mathrm{i}\pi/4$ play the role of keeping the reality of the solution and s is an auxiliary parameter. Note that ρ_n^m can be expressed as $\rho_n^m = 2\partial_s \tau_n(k,l)$ because the auxiliary parameter s works on elements of the above determinant by $2\partial_s \psi_i^{(n)}(k,l) = \psi_i^{(n-1)}(k,l)$. In the lattice KdV and lattice Boussinesq equations, one of τ-functions is also expressed by the derivative of another τ-function with respect to an auxiliary parameter [26, 27]. This is a common property of discrete soliton equations which are directly connected to the Bäcklund transformations of continuous soliton equations.

Let us consider equations (4.20) and (4.21) again. Rewriting equations (4.20) and (4.21), we have

$$\left(\frac{u_{k+1,l} - u_{k,l}}{2} \right)^2 + \delta_{k,l}^2 = a^2, \tag{4.35}$$

$$\left(\frac{u_{k,l+1} + u_{k,l}}{2} \right)^2 + \left(X_{k,l+1} - X_{k,l} + \frac{1}{b} \right)^2 = \frac{1}{b^2}. \tag{4.36}$$

These equations actually give conserved quantities because a^2 and $1/b^2$ are constants.

Introducing

$$I_{k,l} \equiv \left(\frac{u_{k+1,l} - u_{k,l}}{2} \right)^2 + \delta_{k,l}^2, \tag{4.37}$$

$$J_{k,l} \equiv \left(\frac{u_{k,l+1} + u_{k,l}}{2} \right)^2 + \left(X_{k,l+1} - X_{k,l} + \frac{1}{b} \right)^2, \tag{4.38}$$

equations (4.35) and (4.36) imply the following conserved quantities:

$$I_{k,l} = a^2, \qquad J_{k,l} = \frac{1}{b^2}, \tag{4.39}$$

for arbitrary integer values of k and l. Hence, we have

$$I_{k,l+1} - I_{k,l} = 0, \qquad J_{k+1,l} - J_{k,l} = 0. \tag{4.40}$$

A substitution of the corresponding conserved quantities leads to

$$\left(\frac{u_{k+1,l+1} + u_{k+1,l} - u_{k,l+1} - u_{k,l}}{2} \right) \left(\frac{u_{k+1,l+1} - u_{k+1,l} - u_{k,l+1} + u_{k,l}}{2} \right)$$
$$= -(X_{k+1,l+1} + X_{k+1,l} - X_{k,l+1} - X_{k,l})(X_{k+1,l+1} - X_{k+1,l} - X_{k,l+1} + X_{k,l}), \tag{4.41}$$

$$\left(\frac{u_{k+1,l+1} + u_{k+1,l} + u_{k,l+1} + u_{k,l}}{2} \right) \left(\frac{u_{k+1,l+1} + u_{k+1,l} - u_{k,l+1} - u_{k,l}}{2} \right)$$
$$= -\left(X_{k+1,l+1} - X_{k+1,l} + X_{k,l+1} - X_{k,l} + \frac{2}{b} \right)(X_{k+1,l+1} - X_{k+1,l} - X_{k,l+1} + X_{k,l}). \tag{4.42}$$

We can readily show that the difference of equations (4.42) and (4.41) gives equation (4.29), whereas, the quotient is nothing but equation (4.30). In summary, equations (4.35) and (4.36), which imply conserved quantities, can also be derived from the full-discrete analogue of the SP equations (4.29) and (4.30).

5. Conclusions

In this paper, we proposed integrable semi-discrete and full-discrete analogues of the short pulse equation. The *N*-soliton solutions of both the continuous and discrete SP equations were formulated in the form of Casorati determinants, which include multi-loop soliton and multi-breather solutions. Based on the semi-discrete SP equation, a self-adaptive moving mesh method is proposed and used for the numerical solutions of the SP equation. The examples of one- and two-loop soliton solutions show the potential of this novel method for the numerical study of the short pulse equation.

References

[1] Schäfer T and Wayne C E 2004 *Physica* D **196** 90–105
[2] Chung Y, Jones C K R T, Schäfer T and Wayne C E 2005 *Nonlinearity* **18** 1351–74
[3] Robelo M L 1989 *Stud. Appl. Math.* **81** 221–48
[4] Beals R, Rabelo M and Tenenblat K 1989 *Stud. Appl. Math.* **81** 125–51
[5] Sakovich A and Sakovich S 2005 *J. Phys. Soc. Japan* **74** 239–41
[6] Brunelli J C 2005 *J. Math. Phys.* **46** 123507
[7] Brunelli J C 2006 *Phys. Lett.* A **353** 475–8
[8] Sakovich A and Sakovich S 2006 *J. Phys. A: Math. Gen.* **39** L361–367
[9] Kuetche V K, Bouetou T B and Kofane T C 2007 *J. Phys. Soc. Japan* **76** 024004
[10] Kuetche V K, Bouetou T B and Kofane T C 2007 *J. Phys. A: Math. Theor.* **40** 5585–96
[11] Matsuno Y 2007 *J. Phys. Soc. Japan* **76** 084003
[12] Levi D and Ragnisco O (eds) 1998 *SIDE III–Symmetries and Integrability of Difference Equations (CRM Proc. and Lecture Notes* vol 25*)* (Montreal: AMS)
[13] Grammaticos B, Kosmann-Schwarzbach Y and Tamizhmani T (eds) 2004 *Discrete Integrable Systems (Lecture Notes in Physics* vol 644*)* (Berlin: Springer)
[14] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach (Progress in Mathematics* vol 219*)* (Basel: Birkháuser)
[15] Bobenko A I and Suris Y B 2008 *Discrete Differential Geometry (Graduate Studies in Mathematics* vol 98*)* (RI: AMS)
[16] Ohta Y, Maruno K and Feng B F 2008 *J. Phys. A: Math. Theor.* **41** 355205
[17] Feng B F, Ohta Y and Maruno K 2009 arXiv:0905.2693
[18] Hirota R 1972 *J. Phys. Soc. Japan* **33** 1459–63
[19] Mikhailov A V 1979 *JETP Lett.* **30** 414–8
[20] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785–91
[21] Hirota R, Ito M and Kako F 1988 *Prog. Theor. Phys. Suppl.* **94** 42–58
[22] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
[23] Ohta Y, Kajiwara K, Matsukidaira J and Satsuma J 1993 *J. Math. Phys.* **34** 5190–204
[24] Bäcklund A V 1880 *Math. Ann.* **17** 285–328
[25] Hirota R 1977 *J. Phys. Soc. Japan* **43** 2079–86
[26] Kajiwara K and Ohta Y 2008 *J. Phys. Soc. Japan* **77** 054004
[27] Maruno K and Kajiwara K 2009 arXiv:0908.1800